Methodology for Decomposition into Transport and Diffusive Subdomains for the LD Method

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Outline

1. Introduction
   - Background

2. Elements of Methodology
   - Metrics of Transport Effects
   - Domain Decomposition

3. Numerical Results
   - Test A
   - Test B

4. Conclusion
   - Summary
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Highly Diffusive Domains

- A large class of radiative transfer problems contain highly diffusive regions.
- A highly diffusive region is characterized by a large optical thickness and small absorption.
- The diffusion ($P_1$) approximation is valid in such domains.
- It is possible to reduce computational costs by solving a diffusion problem in diffusive subdomains.
The basic idea is to apply domain decomposition methods that
- split the problem domain into transport and diffusion subregions,
- define interface conditions that couple subregions of two different kinds,
- employ hybrid algorithms to solve the problem in each subdomain.

The goal is to approximate the original numerical transport solution on a given spatial-angular grid with minimal error.

Publications
- M. Yavuz & E. Larsen (*TTSP*, 1989)
Current Work

- We consider 1D slab geometry transport problems with isotropic scattering.
- The methodology is developed and applied to the Linear Discontinuous method.
- Transport effects are evaluated by means of metrics based on the quasidiffusion (Eddington) factors.
- The second moment method is used as a basis for the developed methodology.
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**$P_1$ Approximation**

**Moment equations**

\[ \frac{d\phi_1}{dx} + \sigma_a \phi_0 = q , \]
\[ \frac{2}{3} \frac{d\phi_2}{dx} + \frac{1}{3} \frac{d\phi_0}{dx} + \sigma_t \phi_1 = 0 , \]
\[ \phi_n = \int_{-1}^{1} P_n(\mu) \psi(x, \mu) d\mu , \]

**$P_1$ equations**

\[ \frac{d\phi_1}{dx} + \sigma_a \phi_0 = q , \]
\[ \frac{1}{3} \frac{d\phi_0}{dx} + \sigma_t \phi_1 = 0 . \]

\[ M_1(x) \overset{\text{def}}{=} 2 \left| \frac{d\phi_2}{dx} / \frac{d\phi_0}{dx} \right| \ll 1 . \]
\[ d\phi_1 \frac{d\phi_1}{dx} + \sigma_a \phi_0 = q , \]  
\[ \frac{2}{3} \frac{d\phi_2}{dx} + \frac{1}{3} \frac{d\phi_0}{dx} + \sigma_t \phi_1 = 0 , \]  
\[ \phi_n = \int_{-1}^{1} P_n(\mu) \psi(x, \mu) d\mu , \]  
\[ M_1(x) \overset{\text{def}}{=} 2 \left| \frac{d\phi_2}{dx} \right| \left| \frac{d\phi_0}{dx} \right| \ll 1 . \]
Metrics of Transport Effects

$P_1$ Approximation

Moment equations

$$\frac{d\phi_1}{dx} + \sigma_a \phi_0 = q,$$

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Metrics of Transport Effects

Metrics Based on QD (Eddington) Factor

Low-order quasidiffusion (QD) equations

\[
\frac{d\phi_1}{dx} + \sigma_a \phi_0 = q,
\]
\[
\frac{dE[\psi]\phi_0}{dx} + \sigma_t \phi_1 = 0,
\]
\[
E[\psi] = \frac{\int_{-1}^{1} \mu^2 \psi \, d\mu}{\int_{-1}^{1} \psi \, d\mu}.
\]
Metrics Based on QD (Eddington) Factor

Low-order quasidiffusion (QD) equations

\[ \frac{d\phi_1}{dx} + \sigma_a \phi_0 = q, \]

\[ E \frac{d\phi_0}{dx} + \frac{dE}{dx} \phi_0 + \sigma_t \phi_1 = 0, \]

\[ E[\psi] = \int_{-1}^{1} \mu^2 \psi d\mu / \int_{-1}^{1} \psi d\mu. \]

If there are no transport effects, then

\[ E(x) = \frac{1}{3}, \quad \frac{dE(x)}{dx} = 0, \]

\[ M_2(x) \overset{\text{def}}{=} \left| E(x) - \frac{1}{3} \right|, \quad M_3(x) \overset{\text{def}}{=} \left| \frac{dE(x)}{dx} \right|. \]
Metrics of Transport Effects

Metrics Based on QD (Eddington) Factor

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E[\psi] = \int_{-1}^{1} \mu^2 \psi d\mu \bigg/ \int_{-1}^{1} \psi d\mu.
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Second Moment (SM) Method

A transport sweep (high-order equation)

\[
\mu \frac{\partial \psi^{(s+1/2)}}{\partial x} + \sigma_t \psi^{(s+1/2)} = \frac{1}{2} \sigma_s \phi(s) + \frac{1}{2} q.
\]

Low-order SM equations

\[
\frac{dJ^{(s+1)}}{dx} + \sigma_a \phi^{(s+1)} = q,
\]

\[
\frac{1}{3} \frac{d\phi^{(s+1)}}{dx} + \sigma_t J^{(s+1)} = \frac{d}{dx} F^{(s+1/2)}.
\]

The closure term

\[
F^{(s+1/2)} = \int_{-1}^{1} \left( \frac{1}{3} - \mu^2 \right) \psi^{(s+1/2)} d\mu.
\]
Domain Decomposition

Algorithm

Transport Subdomain

\[ L \psi^{(s+1/2)} = \frac{1}{2} \sigma_s \phi^{(s)} + \frac{1}{2} q , \]

\[ F^{(s+1/2)} = \int_{-1}^{1} \left( \frac{1}{3} - \mu^2 \right) \psi^{(s+1/2)} d\mu. \]

Diffusion Subdomain

\[ F^{(s+1/2)} = 0. \]

Whole problem domain

\[ \frac{dJ^{(s+1)}}{dx} + \sigma_a \phi^{(s+1)} = q, \]

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**Transport Subdomain**

\[ L_{\psi(s+1/2)} = \frac{1}{2} \sigma_s \phi(s) + \frac{1}{2} q , \]

\[ F(s+1/2) = \int_{-1}^{1} \left( \frac{1}{3} - \mu^2 \right) \psi(s+1/2) d\mu. \]

**Diffusion Subdomain**

\[ F(s+1/2) = 0. \]

**Whole problem domain**

\[ \frac{dJ^{(s+1)}}{dx} + \sigma_a \phi^{(s+1)} = q , \]

\[ \frac{1}{3} \frac{d\phi(s+1)}{dx} + \sigma_t J^{(s+1)} = \frac{d}{dx} F(s+1/2). \]
We define the angular flux coming into a transport subregion from a neighboring diffusion subregion for the high-order (transport) equation.

- $P_1$ approximation

$$\psi(x, \mu) = \frac{1}{2} \phi + \frac{3}{2} \mu J.$$ 

- Asymptotic interface conditions in discrete form

$$\psi(x, \mu) = \left[ \frac{1}{2} \phi - \frac{3}{2} \mu \left( \frac{1}{3\sigma_t} \right) \frac{d\phi}{dx} \right]_h.$$ 

- No interface conditions are needed for the low-order equations.
Interface Conditions for the High-Order Equation

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- No interface conditions are needed for the low-order equations.
Domain Decomposition

Linear Discontinuous Method

LD equations

\[
\mu m(\psi_{m,i+1/2} - \psi_{m,i-1/2}) + \sigma_{t,i} \Delta x_i \psi_{m,i} = \frac{\Delta x_i}{2} (\sigma_{s,i} \phi_i + q_i),
\]

\[
\mu m \theta_i (\psi_{m,i+1/2} + \psi_{m,i-1/2} - 2\psi_{m,i}) + \sigma_{t,i} \Delta x_i \hat{\psi}_{m,i} = \frac{\Delta x_i}{2} (\sigma_{s,i} \hat{\phi}_i + \hat{q}_i),
\]

\[
\psi_{m,i\mp1/2} = \begin{cases} 
\psi_{m,i} - \hat{\psi}_i, & \mu m < 0, \\
\psi_{m,i} + \hat{\psi}_i, & \mu m > 0.
\end{cases}
\]

Asymptotic expansion of the LD solution in the interior of a diffusive domain leads to the following interface condition:

\[
\psi_{m,i+1/2} = \begin{cases} 
\frac{1}{2} \left( \phi_i + \hat{\phi}_i \right) - \frac{\mu m}{2 \sigma_{t,i} \Delta x_i} (\phi_{i+1/2} - \phi_{i-1/2}), & \mu m > 0, \\
\frac{1}{2} \left( \phi_i + 1 - \hat{\phi}_{i+1} \right) - \frac{\mu m}{2 \sigma_{t,i+1} \Delta x_{i+1}} (\phi_{i+3/2} - \phi_{i+1/2}), & \mu m < 0.
\end{cases}
\]
Domain Decomposition

Linear Discontinuous Method

LD equations

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\mu_m(\psi_{m,i+1/2} - \psi_{m,i-1/2}) + \sigma_{t,i}\Delta x_i \psi_{m,i} = \frac{\Delta x_i}{2}(\sigma_{s,i}\phi_i + q_i),
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\frac{1}{2} \left( \phi_{i+1} - \hat{\phi}_{i+1} \right) - \frac{\mu_m}{2\sigma_{t,i+1}\Delta x_{i+1}} (\phi_{i+3/2} - \phi_{i+1/2}), & \mu_m < 0.
\end{cases}
\]
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Test A

Definition of the Test

<table>
<thead>
<tr>
<th>Region</th>
<th>$0 \leq x \leq 1$</th>
<th>$1 \leq x \leq 11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_t$</td>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$q$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>BC's</td>
<td>$\psi</td>
<td>_{x=0}=1$</td>
</tr>
</tbody>
</table>

- Double $S_4$ Gauss-Legendre quadrature set
- Pointwise convergence criterion with its parameter $= 10^{-15}$
Test A

**Metrics**

![Graph 1](image1.png)

- **Converged**
  - $M_{1,i}$
  - $M_{2,i}$
  - $M_{3,i}$

![Graph 2](image2.png)

- **Estimated**
  - $M_{1,i}$
  - $M_{2,i}$
  - $M_{3,i}$
Test A

Metrics

\[ \varepsilon = 10^{-6} \]

\[ \varepsilon = 10^{-9} \]
Metrics

\[ \varepsilon = 10^{-6} \]
\[ 3 \leq x \leq 10 \]

\[ \varepsilon = 10^{-9} \]
\[ 4 \leq x \leq 9 \]
Numerical Solution with Domain Decomposition

Test A

Relative Error

$x = 10^{-6} [P_1]$

$ε = 10^{-9} [P_1]$

$ε = 10^{-6} [Asy.]$

$ε = 10^{-9} [Asy.]$
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Definition of the Test

<table>
<thead>
<tr>
<th>Region</th>
<th>$0 \leq x \leq 10$</th>
<th>$10 \leq x \leq 30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_t$</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0.5</td>
<td>9.9</td>
</tr>
<tr>
<td>$q$</td>
<td>0.5</td>
<td>0.01</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>BC’s</td>
<td>$\psi</td>
<td>_{x=0} = 0$</td>
</tr>
</tbody>
</table>

- Double $S_4$ Gauss-Legendre quadrature set
- Pointwise convergence criterion with its parameter $= 10^{-15}$
Test B

Metrics

Converged

Estimated
Numerical Solution with Domain Decomposition
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We developed a new spatial decomposition method and applied it to a particular transport discretization in 1D. The numerical results showed that this method has high accuracy in approximating the transport solution computed on a given grid.

The accuracy of the solution depends on the definition of a diffusion subdomain.

The metrics of transport effects can be used to determine accurately diffusion subdomains.

The proposed methodology can be applied to different transport methods and extended to multidimensional geometries and multigroup transport problems.