Neutron transport in Molten Salt Reactors

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Neutronics in systems with moving fuel

- Molten salt systems (MSR), containing liquid fuel in motion, have both static and dynamic properties different from those in traditional reactors.

- Solutions in simple models give insight into the physics of such systems.

- In this talk, closed form analytical solutions are derived for both the static and the dynamic equations.

- The results for the dynamic case show the effect of stronger neutronic coupling and a larger domain of validity of the point behaviour.
The Molten Salt Reactor

Molten salt reactor system (MSR)
Footnote to history: Weinberg’s Molten Salt Reactor (HRE-11)

Weinberg’s salt reactor

The molten salt reactor was conceived as a way of propelling aircraft, re-emerged as a potential fast breeder, and four decades later is attracting interest again. By David Fishlock

NUCLEAR ENGINEERING INTERNATIONAL MARCH 2007
One-dimensional core, dimension $H = 2a$, time of passage $\tau_H = H/u$

Extra-core piping, length $L$, time of passage $\tau_L = L/u$

Delayed neutron precursor equation:

$$L \frac{\partial C(x, t)}{\partial t} + u \frac{\partial C(x, t)}{\partial x} = \beta \nu \Sigma_f \phi(x, t) - \lambda C(x, t)$$

$$C(0, t) = C(H, t - \tau_L) e^{-\lambda \tau_L}$$
A one-dimensional model of MSR

Fuel velocity = \( u \)

Core height: \( H \) \[ \tau_c = \frac{H}{u} \] core transit time

External loop: \( L \); \[ \tau_l = \frac{L}{u} \] loop transit time

Total length: \( T = H + L \); \[ \tau = \frac{T}{u} \] total tr. time
Time dependent diffusion equations

\[ \frac{1}{v} \frac{\partial \phi(z,t)}{\partial t} = D \nabla^2 \phi(z,t) + \left[ \nu \sum f (1 - \beta) - \sum a(z,t) \right] \phi(z,t) + \lambda C(z,t) \]

\[ \frac{\partial C(z,t)}{\partial t} + u \frac{\partial C(z,t)}{\partial z} = \beta v \sum f \phi(z,t) - \lambda C(z,t) \]

Boundary conditions:

\[ \phi(z = 0, t) = \phi_0(z = H, t) = 0 \]

\[ C(0, t) = C(H, t - \frac{L}{u}) e^{-\frac{\lambda L}{u}} = C(H, t - \tau_l) e^{-\lambda \tau_l} \]
**Static equations**

\[ D \nabla^2 \phi_0(z) + \left[ \nu \Sigma_f (1-\beta) - \Sigma_a \right] \phi_0(z) + \lambda C_0(z) = 0 \]

\[ u \frac{\partial C_0(z)}{\partial z} - \beta \nu \sum_f \phi_0(z) + \lambda C_0(z) = 0 \]

**Boundary conditions:**

\[ \phi_0(z = 0) = \phi_0(z = H) = 0 \]

\[ C(0) = C(H) e^{-\frac{L}{u}} = C(H) e^{-\lambda \tau_l} \]

Delayed neutron precursors do not disappear from the static equations.
The equation for the neutron noise

\[ D \nabla^2 \delta \phi(z, \omega) + \left[ \nu \Sigma f (1 - \beta) - \Sigma a - \frac{i \omega}{\nu} \right] \delta \phi(z, \omega) + \lambda e^{-\frac{\lambda(\omega)}{u}} \beta \nu \sum f \frac{\lambda(\omega)}{u} = \delta \sum a (z, \omega) \phi_0(z) \equiv S(z, \omega) \]

where

\[ \lambda(\omega) = \lambda + i \omega \]
The Greens function (response to a localised perturbation)

\[ \nabla^2 \delta \phi(z, \omega) + B^2(\omega) \delta \phi(z, \omega) + \lambda e^{-\lambda(\omega) ÷ u} \frac{\beta \nu \Sigma_f}{Du} \times \]

\[ \times \left\{ \begin{array}{l}
\frac{e^{-\lambda(\omega) ÷ u}}{1 - e^{-\lambda(\omega) ÷ u}} \int_0^H e^{\lambda(\omega) ÷ u} \delta \phi(z, \omega) dz + \int_0^z e^{\lambda(\omega) ÷ u} \delta \phi(z', \omega) dz'
\end{array} \right\} \]

\[ = \delta(z - z_0) \]

with

\[ B^2(\omega) = B_0^2 \left( 1 - \frac{i \omega \Lambda}{\rho_\infty - \beta} \right); \quad \lambda(\omega) = \lambda + i \omega \]
Solution of the static equations

Eliminating the precursors by quadrature, one obtains the integro-differential equation

\[ \nabla^2 \phi_0(z) + B_0^2 \phi_0(z) + \]

\[ + e^{-z} \frac{\lambda}{u} \lambda \beta \nu \sum f \left[ \frac{e^{-\lambda \tau}}{1-e^{-\lambda \tau}} \int e^{z'} \frac{\lambda}{u} \phi_0(z')dz' + \int e^{z'} \frac{\lambda}{u} \phi_0(z')dz' \right] = 0 \]

\[ B_0^2 = \frac{\nu \sum f (1 - \beta) - \sum a}{D} < \frac{\pi}{2a} \]

Solution in form of expansion in the eigenfunctions of the traditional problem (Sandra Dulla).
The full problem: Static flux and precursor density

- The flux has the same shape as in a traditional reactor
- But precursor density distribution depends on fuel velocity

**Figure:** Static flux

**Figure:** Neutron precursor density for different velocities
Flux and precursor distributions

Fig. 2. The static flux in an MSR for $u = 50 \text{ cm/s}$

Fig. 3. The distribution of delayed neutron precursors in an MSR for different fuel velocities. (a) $u = 0.1 \text{ cm/s}$, (b) $u = 1 \text{ cm/s}$, (c) $u = 2 \text{ cm/s}$, (d) $u = 10 \text{ cm/s}$, and (e) $u = 50 \text{ cm/s}$
Finite velocity: Static flux solution

- Flux peaks at different points in core depending on velocity

**Figure:** Static flux at different velocities
Criticality, as a function of circulation speed:

<table>
<thead>
<tr>
<th>Case</th>
<th>$u$ [cm/s]</th>
<th>$\tau_l$ [s]</th>
<th>$k_{eff}$</th>
<th>$\Delta \rho$ [pcm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>$\rightarrow \infty$</td>
<td>1.000000</td>
<td>0</td>
</tr>
<tr>
<td>(b)</td>
<td>0.1</td>
<td>1000</td>
<td>0.99997</td>
<td>$-2$</td>
</tr>
<tr>
<td>(c)</td>
<td>1</td>
<td>100</td>
<td>0.99868</td>
<td>$-132$</td>
</tr>
<tr>
<td>(d)</td>
<td>2</td>
<td>50</td>
<td>0.99736</td>
<td>$-265$</td>
</tr>
<tr>
<td>(e)</td>
<td>5</td>
<td>20</td>
<td>0.99587</td>
<td>$-415$</td>
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<tr>
<td>(f)</td>
<td>10</td>
<td>10</td>
<td>0.99543</td>
<td>$-459$</td>
</tr>
<tr>
<td>(g)</td>
<td>50</td>
<td>2</td>
<td>0.99526</td>
<td>$-476$</td>
</tr>
<tr>
<td>(h)</td>
<td>$\rightarrow \infty$</td>
<td>$\rightarrow 0$</td>
<td>0.99526</td>
<td>$-476$</td>
</tr>
</tbody>
</table>
Simplification: infinite fuel speed

\[ \nabla^2 \phi_0(z) + B_0^2 \phi_0(z) \]

\[ + \left( -z \frac{\lambda}{u} \frac{\lambda \beta \nu \Sigma_f}{Du} \int_0^H e^{-\lambda \tau} \frac{\lambda}{u} \phi_0(z') \, dz' + \int_0^H e^{-\lambda \tau} \frac{\lambda}{u} \phi_0(z') \, dz' \right) = 0 \]

For \( u = \infty (\tau = 0) \):

\[ \nabla^2 \phi(z) + B_0^2 \phi(z) + \frac{\beta \nu \Sigma_f}{DT} \int_0^H \phi(z') \, dz' = 0 \]

Analytical solutions exist for both the static and the dynamic problem
Static equation

\[ \nabla^2 \phi_0(x) + B_0^2 \phi_0(x) + \frac{\eta_0}{T} \int_{-a}^{a} \phi_0(x')dx' = 0 \]

\[ B_0^2 = \frac{\nu \Sigma_f (1 - \beta) - \Sigma_a}{D} \left( \frac{\pi}{2a} \right)^2 ; \quad \eta_0 = \frac{\nu \Sigma_f \beta}{D} \]

Solution:

\[ \phi_0(x) = A[\cos B_0 x - \cos B_0 a] \]

Criticality equation

\[ B_0^2 \cos B_0 a + \frac{2a \eta_0}{T} \cos B_0 a - \frac{2\eta_0}{TB_0} \sin B_0 a = 0 \]
Dynamic equation: traditional system

\[ \nabla^2 \delta \phi(x, \omega) + B^2(\omega) \delta \phi(x, \omega) = \frac{\delta \Sigma \alpha(x, \omega) \phi_0(x)}{D} \equiv S(x, \omega) \]

where

\[ B^2(\omega) = B_0^2 \left( 1 - \frac{1}{\rho \infty G_0(\omega)} \right); \quad B_0 = \frac{\pi}{2a} \]

Solution: Green's function

\[ \nabla^2 G(x, x_0, \omega) + B^2(\omega) G(x, x_0, \omega) = \delta(x - x_0) \]

\[ \delta \phi(x, \omega) = \int_{-a}^{a} G(x, x_0, \omega) S(x_0, \omega) \, dx_0 \]
Solution

\[ G(x, x_0, \omega) = \]

\[ -\frac{1}{B(\omega) \sin 2B(\omega)a} \begin{cases} 
\sin B(\omega)(a + x) \sin B(\omega)(a - x_0) & x \leq x_0 \\
\sin B(\omega)(a - x) \sin B(\omega)(a + x_0) & x > x_0 
\end{cases} \]
Simplification to $u = \infty$

$$\nabla^2 G(x, x_0, \omega) + B^2(\omega)G(x, x_0, \omega)$$

$$+ \frac{\eta(\omega)}{T} \int_{-a}^{a} G(x, x_0, \omega) dx' = \delta(x - x_0)$$

with

$$B^2(\omega) = B_0^2 \left( 1 - \frac{i\omega \Lambda}{\rho_\infty - \beta} \right); \quad B_0^2 < \frac{\pi}{2a}; \quad \eta(\omega) = \frac{\lambda}{\lambda + i\omega} \eta_0$$
Solution

\[ G(x, x_0, \omega) = \frac{\eta(\omega)\phi_0(x, \omega)\phi_0(x_0, \omega)}{A^2TK(\omega)B(\omega)^2 \cos B(\omega)a} \]

\[-\frac{1}{B(\omega)\sin 2B(\omega)a} \begin{cases} 
\sin B(\omega)(a - x_0)\sin B(\omega)(a + x) & x < x_0 \\
\sin B(\omega)(a + x_0)\sin B(\omega)(a - x) & x > x_0 
\end{cases} \]

with

\[ \phi_0(x, \omega) = A[\cos B(\omega)x - \cos B(\omega)a] \]

\[ K(\omega) = B^2(\omega)\cos B(\omega)a + \frac{2a\eta(\omega)}{T}\cos B(\omega)a - \frac{2\eta(\omega)}{TB(\omega)}\sin B(\omega)a \]
New developments

• The case of infinite fuel velocity was used - to investigate the point kinetic behaviour at low frequencies (ANS Annual Meeting 2011 Florida) - the neutronic response to various perturbations (PHYSOR 2010 Pittsburgh)

• However, the case of infinite fuel velocity does not allow to study the effect of varying fuel velocity

• Hence further approximations were searched for

• It then turned out that the full problem has a compact analytical solution
Physical meaning of $u = \infty$ and of the integral terms

$$\nabla^2 \phi_0(z) + B_0^2 \phi_0(z)$$

$$+ e^{-z\lambda u} \frac{\lambda \beta \nu \sum f}{Du} \left( \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \int_0^H e^{u \phi_0(z')} dz' + \int_0^z e^{u \phi_0(z')} dz' \right) = 0$$

$$C_0(z) = e^{-z\lambda u} \frac{\beta \nu \sum f}{u} \left( \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \int_0^H e^{u \phi_0(z')} dz' + \int_0^z e^{u \phi_0(z')} dz' \right)$$
Comparison with the traditional case

\[
\frac{\partial C(z, t)}{\partial t} = \beta \nu \sum_f \phi(z, t) - \lambda C(z, t)
\]

\[
C(z, t) = \beta \nu \sum_f \int_{-\infty}^{t} e^{-\lambda(t-t')} \phi(z, t) dt'
\]

In the stationary (time-independent) case:

\[
C_0(z) = \beta \nu \sum_f \int_{-\infty}^{t} e^{-\lambda(t-t')} \phi_0(z) dt'
\]
Neutrons generated at time $t' < t$ were born at

$$z' = z - u(t - t')$$

$$dt' = dz'/u; \quad t - t' = \frac{z - z'}{u}$$

$$C_0(z) = \frac{\beta \nu \Sigma_f}{u} \int_{-\infty}^{z} e^{-\frac{\lambda}{u}(z-z')} \phi_0(z')dz'$$
Moving precursors: finite slab

\[ C_0(z) = e^{-\lambda u / z} \frac{\beta \nu \sum_f}{u} \left( \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \int_0^H e^u \phi_0(z')dz' + \int_0^z e^u \phi_0(z')dz' \right) \]

\[ \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} = e^{-\lambda \tau} + e^{-2\lambda \tau} + e^{-3\lambda \tau} \ldots \]

\[ C_0(z) = \frac{\beta \nu \sum_f}{u} \left( \sum_{n=1}^{\infty} \int_0^H e^{-\frac{\lambda}{u}(z'-z)-n\lambda \tau} \phi_0(z')dz' + \int_0^z e^{-\frac{\lambda}{u}(z'-z)} \phi_0(z')dz' \right) \]

The different terms in the sum correspond to the once, twice, three times recirculated precursors
The dynamic case

\[
\nabla^2 \delta \phi(z, \omega) + B^2(\omega) \delta \phi(z, \omega) + \lambda e^{\frac{-\lambda(\omega)}{u}} \frac{\beta \nu \Sigma_f}{Du} \times \\
\left\{ \frac{e^{-\lambda(\omega)\tau}}{1 - e^{-\lambda(\omega)\tau}} \int_0^H e^{\frac{\lambda(\omega)}{u}} \delta \phi(z, \omega)dz + \int_0^z e^{\frac{\lambda(\omega)}{u}} \delta \phi(z', \omega)dz' \right\} \\
= \delta \Sigma_a(z, \omega) \phi_0(z) \equiv S(z, \omega)
\]

\[
\delta C(z, \omega) = e^{\frac{-(\lambda+i\omega)}{u}} \frac{\beta \nu \Sigma_f}{u} \left\{ \frac{e^{-(\lambda+i\omega)\tau}}{1 - e^{-(\lambda+i\omega)\tau}} \int_0^H e^{\frac{(\lambda+i\omega)}{u}} \delta \phi(z', \omega)dz' \right\} + \int_0^z e^{\frac{\lambda+i\omega}{u}} \delta \phi(z', \omega)dz' \equiv \delta C_1(z, \omega) + \delta C_2(z, \omega)
\]
The integral terms in the time domain

\[ \delta C_2(z, \omega) = \frac{\beta \nu \sum f}{u} \int_0^z e^{-i \omega (z-z')} e^{-\frac{\lambda}{u} (z-z')} \delta \phi(z', \omega) dz' \]

After inverse Fourier transform:

\[ \delta C_2(z, t) = \frac{\beta \nu \sum f}{u} \int_0^z e^{-\frac{\lambda}{u} (z-z')} \delta \phi(z', t - \frac{z - z'}{u}) dz' \]

Similarly, for the first integral one obtains

\[ \delta C_1(z, t) = \frac{\beta \nu \sum f}{u} \sum_{n=1}^{\infty} \int_0^H e^{-\frac{\lambda}{u} (z-z') - n \lambda \tau} e^{-\frac{\lambda}{u} (z - z') - n \tau} \delta \phi(z', t - \frac{z - z'}{u} - n \tau) dz' \]
The approximation of no recirculation (long recirculation time)

\[ C_0(z) = e^{-\frac{\lambda z}{u}} \frac{\beta \nu \Sigma_f}{u} \left( \frac{e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \int_0^H e^{u \phi_0(z')} dz' + \int_0^z e^{u \phi_0(z')} dz' \right) \]

Now the first integral is neglected besides the second, leading to

\[ \nabla^2 \phi_0(z) + B_0^2 \phi_0(z) + \frac{\lambda \beta \nu \Sigma_f}{Du} \int_0^z e^{-\frac{\lambda (z-z')}{u}} \phi_0(z') dz' = 0 \]

This equation has a closed form analytical solution.
Solution

\[ \phi'''(z) + \frac{\lambda}{u} \phi''(z) + B_0^2 \phi'(z) + \frac{\lambda}{u} \left( B_0^2 + \frac{\beta \nu \Sigma_f}{D} \right) \phi(z) = 0 \]

Characteristic equation:

\[ k^3 + \frac{\lambda}{u} k^2 + B_0^2 k + \frac{\lambda}{u} \left( B_0^2 + \frac{\beta \nu \Sigma_f}{D} \right) = 0 \]

On physical grounds we expect

\[ k_{1,2} = \alpha \pm i \beta; \quad k_3 = \gamma \]
Solution

\[ \phi_0(z) = A_1 e^{\alpha z} \sin(\beta z) + A_2 e^{\alpha z} \cos(\beta z) + A_3 e^{\gamma z} \]

Two coefficients can be eliminated by the boundary conditions:

\[ \phi_0(z) = A[e^{\alpha z} \sin \beta z(e^{\gamma H} - e^{\alpha H} \cos \beta H) \]
\[ - e^{\alpha H} \sin \beta H(e^{\gamma z} - e^{\alpha z} \cos \beta z)] \]

Or, in the x-coordinate system, in the reactor centre:

\[ \phi_0(x) = A\{e^{\alpha x} \cos(\beta x) - e^{-(\alpha-\gamma)a} \cos(\beta a)e^{\gamma x} - \]
\[ - \cot(\beta a) \tanh[(\alpha - \gamma)a][e^{\alpha x} \sin(\beta x) + e^{-(a-\gamma)a} \sin(\beta a)e^{\gamma x}]) \]
Substituting back to the original equation gives the criticality condition. This can be written symbolically as

\[ 0 = \sum_{n=1}^{3} \frac{A_n}{\alpha_n + \lambda / u} \]

In reality this is much more complicated, because the relationship between the \( A_n \) has to be used explicitly.
The full integro-differential equation leads to exactly the same differential equation as the one with no recirculation. Hence it has the same characteristic equation and same form of the solution, only the criticality condition is different (because it is determined by the full original integro-differential equation):

\[
0 = \sum_{n=1}^{3} \frac{A_n}{\alpha_n + \lambda / u} \left(-1 + \frac{1}{e^{\lambda \tau} - 1} \left(e^{(\alpha_n + \lambda / u)H} - 1\right)\right)
\]
Reverting to the case of infinite velocity

\[ k^3 + \frac{\lambda}{u} k^2 + B_0^2 k + \frac{\lambda}{u} (B_0^2 + \frac{\beta \nu \Sigma_f}{D}) = 0 \]

\[ \Rightarrow k^3 + B_0^2 k = 0; \quad \alpha = \gamma = 0; \quad \beta = B_0 \]

Then the full solution will revert to that obtained before

\[ \phi_0(x) = A \{ e^{\alpha x} \cos(\beta x) - e^{-(\alpha-\gamma)a} \cos(\beta a) e^{\gamma x} - \]

\[ - \cot(\beta a) \tanh[(\alpha - \gamma)a] \left[ e^{\alpha x} \sin(\beta x) + e^{-(a-\gamma)a} \sin(\beta a) e^{\gamma x} \right] \}

\[ \Rightarrow \phi_0(x) = A (\cos B_0 x - \cos B_0 a) \]
Static solutions for zero flux and logarithmic boundary conditions

For the general case, the two solutions coincide
Static solutions for zero flux and logarithmic boundary conditions

For the no recirculation case, and with high beta-eff, the two solutions differ significantly.
Dynamic behaviour: The Green’s function

The point kinetic behaviour is retained up to higher frequencies (or system sizes) as in an equivalent traditional system.

Figure: $\omega = 0.01 \text{ rad/s}$

Figure: $\omega = 1 \text{ rad/s}$
Dynamic behaviour: The Green’s function (cont)

The physical reason is the spatial coupling, represented by the moving precursors and the smaller value of beta-eff.

**Figure:** $\omega = 100 \text{ rad/s}$

**Figure:** $\omega = 1000 \text{ rad/s}$
Conclusions

• The MSR equations in one-group diffusion theory have a closed form analytic solution for both the static and the dynamic case.

• The infinite fuel velocity (short recirculation time) and no recirculation are limiting cases with even simpler analytical solutions, which are useful conceptual models for analytical investigations.