Quantum corrections on the radiative transfer equation

J. Rosato

Laboratoire PIIM, Université de Provence / CNRS, Marseille, France

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The radiative transfer equation (RTE)

\[ \frac{1}{c} \frac{\partial I}{\partial t} + \vec{n} \cdot \vec{\nabla} I = \text{r.h.s.} \]

1) Can the RTE be obtained from quantum mechanics?

2) If yes, does such a derivation provide new physics?
A first, heuristic approach

a) Define the photons as point-like particles evolving in phase space

b) Associate them with QM-particles, using the correspondence principle

Then, a QM-version of the RTE should be obtained…
The correspondence principle

i) \((\vec{r}_1(t)\ldots\vec{r}_N(t), \vec{p}_1(t)\ldots\vec{p}_N(t)) \rightarrow (\hat{\vec{r}}_1\ldots\hat{\vec{r}}_N, \hat{\vec{p}}_1\ldots\hat{\vec{p}}_N)\)

\[ [\hat{x}_i, \hat{p}_{x_1}] = \hat{x}_i\hat{p}_{x_1} - \hat{p}_{x_1}\hat{x}_i = i\hbar \]

\[ \ldots \]

ii) Density operator \(\hat{\rho}(t)\) \quad \langle \ldots \rangle = \text{Tr}(\hat{\rho} \ldots)\)

Heuristic definition of a quantum phase space distribution:

\[
f(\vec{r}, \vec{p}, t) = \text{Tr}\left( \hat{\rho}(t) \sum_{j=1}^{N} \delta(\vec{r} - \hat{\vec{r}}_j)\delta(\vec{p} - \hat{\vec{p}}_j) \right)\]

\[
i\hbar \frac{d\hat{\rho}}{dt}(t) = [\hat{H}, \hat{\rho}(t)]\]

Liouville – von Neumann equation
The photon localization problem

Photon = particle of mass 0 & spin 1

symmetry considerations

No position operator satisfying the canonical commutation rules exists

Pryce (1948)
Newton and Wigner (1949)
Rosewarne and Sarkar, Quantum Opt. (1992)

\[
f(\vec{r}, \vec{p}, t) \neq \text{Tr}\left(\hat{\rho}(t) \sum_{j=1}^{N} \delta(\vec{r} - \hat{r}_j)\delta(\vec{p} - \hat{p}_j)\right)
\]

Need of a more sophisticated approach
Second quantization

QED: in vacuum, the radiation field is equivalent to an infinite set of independent quantum harmonic oscillators

\[ \hat{H}_R = \sum_j \hbar \omega_j \left( \hat{N}_j + \frac{1}{2} \right) \]

\[ \hat{N}_j |n_j\rangle = n_j |n_j\rangle \]

\[ \hat{N}_j = \hat{a}_j^+ \hat{a}_j \]

A photon is characterized by \( j \equiv (\vec{k}, \varepsilon) \)

A physical state with \( n \) photons of mode \( j \) is described by \( |n_j\rangle \)

If atoms are present, \( \hat{H} = \hat{H}_{at} + \hat{H}_R + \hat{V} \)
Quantum phase space distribution function

Phase space number operator (Klimontovich, 1957)

\[
\hat{N}_{\epsilon, \vec{k}}(\vec{r}, t) = \frac{1}{\pi^3} \int d^3 k' \hat{a}_{\epsilon, \vec{k}'}^+ \hat{a}_{\epsilon, \vec{k} + \vec{k}'} e^{i \varphi_{\epsilon, \vec{k}}(\vec{r}, t)}
\]

with \[ \varphi_{\epsilon, \vec{k}}(\vec{r}, t) = 2 \vec{k'} \cdot \vec{r} + c t \left( |\vec{k} - \vec{k}'| - |\vec{k} + \vec{k}'| \right) \]

\[
W(\vec{r}, \vec{k}, t) = \sum \text{Tr} \left( \hat{\rho}(t) \hat{N}_{\epsilon, \vec{k}}(\vec{r}, t) \right)
\]

Photon Wigner distribution
Quantum transport equation

\[ \left( \frac{\partial}{\partial t} + c \frac{\vec{k}}{k} \cdot \vec{\nabla} \right) W(\vec{r}, \vec{k}, t) \approx \sum_{\tilde{\varepsilon}} \left( \frac{\partial}{\partial t} + c \frac{\vec{k}}{k} \cdot \vec{\nabla} \right) \sum_{\tilde{\varepsilon}} \text{Tr} \left( \hat{\rho}(t) \hat{N}_{\tilde{\varepsilon}, \tilde{\varepsilon}}(\vec{r}, t) \right) \]

\[ \approx \sum_{\tilde{\varepsilon}} \text{Tr} \left( \frac{d\hat{\rho}}{dt}(t) \hat{N}_{\tilde{\varepsilon}, \tilde{\varepsilon}}(\vec{r}, t) \right) \]

at first order in \(\lambda/\lambda_{\text{mfp}}\)

\[ \frac{d\hat{\rho}}{dt}(t) = -\frac{1}{\hbar^2} \int_0^t d\tau [\hat{V}(t-\tau), \hat{\rho}(t-\tau)] \]

\[ \approx -\frac{1}{\hbar^2} \int_0^\infty d\tau [\hat{V}(t-\tau), \hat{\rho}(t)] \]

Master equation

Spatial coherence

\[ \lambda = \frac{h}{p} = \frac{c}{\omega} \]

\[ \lambda_{\text{coherence}} = \frac{h}{\Delta p} = \frac{c}{\Delta \omega} \]
Coherence effects in magnetic fusion

ITER divertor, “high”-density conditions: $N_{\text{at}} > 10^{14} \text{ cm}^{-3}$

Need to solve the quantum RTE
The quantum RTE has been solved in 1D by Fourier series.

(a) $\frac{\lambda_c}{\lambda_{mfp}} = 0$

(b) $\frac{\lambda_c}{\lambda_{mfp}} = 10$

Distortion of the Wigner function
Spectral profile of Ly-\(\alpha\)

The signal is larger than expected from the conventional RTE
Escape factor

\[ N_2 A_{21} - N_1 B_{12} \bar{I} \equiv N_2 A_{21} \theta \]

Reduction of the plasma opacity

\[ \lambda_c / \lambda_{mfp} = 0 \]
\[ \lambda_c / \lambda_{mfp} = 10 \]
Coherence effects in other plasmas?

\[ \frac{\lambda_{\text{coherence}}}{\lambda_{\text{mfp}}} \sim f_{ud} N_d \Delta \omega_{1/2}^{-2} \]

\[ \lambda_{\text{mfp}} < \lambda_c \quad \text{coherence effects} \]

\[ \lambda_{\text{mfp}} > \lambda_c \quad \text{no coherence effects} \]

Inertial fusion
Astrophysics
WDM & HDM

Magnetic fusion (divertors)
A simplified 1-D slab model

Ar-doped D-plasma
- \( N_e = 5 \times 10^{23} \text{ cm}^{-3} \)
- \( T_{e,i} = 1200 \text{ eV} \)

\( W(z = 0) \propto I_0 \)
\( W(z) = -\int_0^z dz' K(z - z')W(z') \)
\( K(z) \propto \frac{1}{l_c} e^{-z/l_c} \)

Absorption due to Ar-Ly-\( \alpha \) (Ar\(^{17+}\), 1s-2p, 3.3 keV)

\[ \begin{align*}
    &l_0 \quad \text{X ray irradiation, large spectral band} \\
    &z = 0 \quad \text{z} \\
    &\text{Transmission} \\
    &2\% \text{ Ar}^{17+} \\
    &\text{without coherence} \quad \text{with coherence}
\end{align*} \]
A proper derivation of a transport equation for photons can be / has been done: **Wigner’s phase space formalism**

It provides the transfer equation commonly presented in the literature in the limit $\lambda_{coherence} \ll \lambda_{mfp}$

The spatial coherence alters the plasma’s opacity => *confirmation both for low- and high-density plasmas*

Comparison to experiments????
References

Radiative transfer

S. Chandrasekhar, *Radiative Transfer* (Dover, New York, 1960)

Photon localization


Quantum transport equation for photons